

# Quantum Jumps on a Circle

K. Kowalski and J. Rembieliński

*Department of Theoretical Physics, University of Łódź, ul. Pomorska 149/153,  
90-236 Łódź, Poland*

It is demonstrated that in contrast to the well-known case with a quantum particle moving freely in a real line, the wave packets corresponding to the coherent states for a free quantum particle on a circle do not spread but develop periodically in time. The discontinuous changes during the course of time in the phase representing the position of a particle can be interpreted as the quantum jumps on a circle.

02.30.Gp, 03.65.-w, 03.65.Sq

In spite of the recent great progress in nanotechnology enabling for example the construction of nanoscopic quantum rings [1], the theory of quantization in the case when the configuration space exhibits a nontrivial topology can hardly be called satisfactory. For example, the coherent states for such a simple system as a quantum particle on a circle have been introduced only recently [2,3]. In this work we discuss the spreading of the wave packets in the case of a free motion of a quantum particle in a circle. Namely, we show that in contrast to the popular example of a free evolution on a real line [4] the wave packets referring to the coherent states for a quantum particle on a circle do not spread over the configuration space but execute regular oscillations. We also demonstrate that such form of the evolution of the wave packets leads to the discontinuous changes in the expectation values of the phase operator i.e. the position observable for a quantum particle on a circle. From the classical point of view such discontinuities can be viewed as the quantum jumps.

We first collect the basic facts about the quantum mechanics on a circle  $S^1$ . Consider a free quantum particle on  $S^1$ . We assume for simplicity that the particle has unit mass and it moves in a unit circle. The classical Hamiltonian is of the form

$$H = \frac{1}{2}J^2, \quad (1)$$

where  $J$  is the angular momentum canonically conjugated to the angle  $\varphi$  i.e. we have the Poisson bracket such that

$$\{\varphi, J\} = 1, \quad (2)$$

leading, accordingly to the rules of the canonical quantization to the commutator

$$[\hat{\varphi}, \hat{J}] = i, \quad (3)$$

where we set  $\hbar = 1$ . Motivated by numerous misinterpretations and misunderstandings which can be met in the literature related with the commutator (3) we now study (3) in a more detail. Consider the Hilbert space  $\mathcal{H}$  of square integrable functions on  $S^1$  satisfying the generalized periodicity condition  $f(\varphi + 2\pi) = e^{2\pi i \lambda} f(\varphi)$ , where  $\lambda$  is fixed and  $0 \leq \lambda < 1$ , specified by the scalar product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\varphi f^*(\varphi) g(\varphi). \quad (4)$$

In the following we shall call the elements of  $\mathcal{H}$  the  $\lambda$ -phase periodic functions. The integral (4) is understood as the Lebesgue integral. It is also important to note that we identify the vectors in  $\mathcal{H}$  equal to one another almost everywhere. More precisely, denoting the set of  $\lambda$ -phase periodic functions with the domain having Lebesgue measure zero by  $N$ , the functions  $f$  and  $f + N$  represent the same vector (equivalence class) in  $\mathcal{H}$ , i.e. we actually have  $\mathcal{H} = L^2(S^1)/N$ . In the following we will represent vectors in  $\mathcal{H}$  by their representatives. Now, the hermitian angular momentum operator  $\hat{J}$  acts on the differentiable functions (representatives of vectors belonging to the domain of  $\hat{J}$ ) in the following way:

$$\hat{J}f(\varphi) = -i \frac{d}{d\varphi} f(\varphi). \quad (5)$$

We remark that there exists the selfadjoint closure of  $\hat{J}$  defined on the set of  $\lambda$ -phase periodic absolutely continuous functions. Of course, the operator of the multiplication by  $\varphi$ , suggested by the naive utilization of (3) and (5), maps a  $\lambda$ -phase periodic function into non- $\lambda$ -phase periodic function. The only exception is the zero function  $f(\varphi) \equiv 0$ . An adequate selfadjoint bounded angle operator preserving the  $\lambda$ -phase periodicity of a function is defined by

$$\hat{\varphi}f(\varphi) = \left( \varphi - 2\pi \left[ \frac{\varphi}{2\pi} \right] \right) f(\varphi), \quad (6)$$

where  $[x]$  is the biggest integer in  $x$ . We remark that an analogous form of the angle operator was indicated in [5]. Now, it is evident that the commutator of  $\hat{J}$  and  $\hat{\varphi}$  is trivially defined in  $\mathcal{H}$ . Indeed, for arbitrary nontrivial  $\lambda$ -phase periodic function  $f \in \mathcal{H}$ ,  $\hat{\varphi}f$  is an element of  $\mathcal{H}$  but due to the discontinuity of  $\hat{\varphi}f$  resulting from (6) it is not an element of the domain of  $\hat{J}$ . We conclude that the canonical commutation relation (3) is defined only on the zero vector. It seems that the most natural solution of this problem is to introduce the unitary operator

$$U = e^{i\hat{\varphi}} \quad (7)$$

satisfying

$$Uf(\varphi) = e^{i\hat{\varphi}}f(\varphi) = e^{i(\varphi - 2\pi[\frac{\varphi}{2\pi}])}f(\varphi) = e^{i\varphi}f(\varphi), \quad (8)$$

where  $\hat{\varphi}$  is given by (6). Now, an immediate consequence of (5) and (8) is the following commutation relation:

$$[\hat{J}, U] = U. \quad (9)$$

Since  $U$  is defined on an arbitrary element of  $\mathcal{H}$  and it leaves invariant the domain of  $\hat{J}$ , therefore, in opposition to (5), the commutation relation (9) is defined on the dense set in  $\mathcal{H}$ . This observation as well as behavior of the expectation values of  $U$  in the coherent state (see figures 1 and 3) suggest that  $U$  is a better candidate to represent the position of a quantum particle on a circle than  $\hat{\varphi}$ . The unitary operator  $U$  satisfying (9) was discussed for the first time by Luisell [6]. We stress that the problem of the description of the angular position should not be mixed up with the problem of the proper definition of the phase operator related to the polar decomposition of the Bose annihilation operator for harmonic oscillator [7].

Now, let  $e_j(\varphi) = e^{ij\varphi}$ , where  $j = k + \lambda$ , and  $k$  is integer, represent the eigenvectors of the angle momentum operator  $\hat{J}$

$$\hat{J}e_j(\varphi) = je_j(\varphi) \quad (10)$$

corresponding to the eigenvalue  $j$ . The operators  $U$  and  $U^\dagger$  act on the vectors  $e_j(\varphi)$  as the ladder operators. Namely

$$Ue_j(\varphi) = e_{j+1}(\varphi), \quad (11a)$$

$$U^\dagger e_j(\varphi) = e_{j-1}(\varphi). \quad (11b)$$

Demanding the time-reversal invariance of the algebra (9) we find that the only possibility left is  $\lambda = 0$  or  $\lambda = 1/2$  [2]. Therefore,  $j$  can be only integer or half-integer. We refer to the case with integer (half-integer)  $j$  as to the boson (fermion) case. Clearly, the vectors  $e_j(\varphi)$  span the Hilbert space of states  $\mathcal{H}$ . We finally point out that the case with  $\lambda = 0$  ( $\lambda = 1/2$ ) refers to periodic (anti-periodic) functions.

Consider now the coherent states for a particle on a circle [2]. These states can be defined as a solution of the eigenvalue equation

$$Zf_\xi(\varphi) = \xi f_\xi(\varphi), \quad (12)$$

where  $Z = e^{-\hat{J} + \frac{1}{2}}U$ , and the complex number  $\xi = e^{-l + i\alpha}$  parametrizes the cylinder which is the classical phase space for the particle moving in a circle. The coherent states are given by

$$f_\xi(\varphi) = \theta_3\left(\frac{1}{2\pi}(\varphi - \alpha - il)\middle|\frac{i}{2\pi}\right), \quad (\text{boson case}) \quad (13a)$$

$$f_\xi(\varphi) = \theta_2\left(\frac{1}{2\pi}(\varphi - \alpha - il)\middle|\frac{i}{2\pi}\right), \quad (\text{fermion case}) \quad (13b)$$

where  $\theta_3$  and  $\theta_2$  are the Jacobi theta-functions defined by

$$\theta_3(v|\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} (e^{i\pi v})^{2n}, \quad (14a)$$

$$\theta_2(v|\tau) = \sum_{n=-\infty}^{\infty} q^{(n-\frac{1}{2})^2} (e^{i\pi v})^{2n-1}, \quad (14b)$$

where  $q = e^{i\pi\tau}$  and  $\text{Im } \tau > 0$ . It is interesting that the above coherent states have the Bargmann representation which is closely related to the general construction introduced by Stenzel [8]. The overlap integral is

$$\langle f_\xi, f_{\xi'} \rangle = \theta_3\left(\frac{1}{2\pi}(\alpha - \alpha') - \frac{l+l'}{2} \frac{i}{\pi} \middle| \frac{i}{\pi}\right), \quad (\text{boson case}) \quad (15a)$$

$$\langle f_\xi, f_{\xi'} \rangle = \theta_2\left(\frac{1}{2\pi}(\alpha - \alpha') - \frac{l+l'}{2} \frac{i}{\pi} \middle| \frac{i}{\pi}\right). \quad (\text{fermion case}) \quad (15b)$$

The expectation value  $\langle \hat{J} \rangle_{f_\xi}$  of  $\hat{J}$  in the normalized coherent state  $f_\xi / \|f_\xi\|$  is

$$\langle \hat{J} \rangle_{f_\xi} \approx l, \quad (16)$$

where we have the exact equality in the case with  $l$  integer or half-integer and the maximal error arising in the case  $l \rightarrow 0$  is of order 0.1%. Therefore, the parameter  $l$  in  $\xi$  can be really identified with the classical angular momentum. Further, we have the following formula on the relative expectation value  $\langle U \rangle_{f_\xi} / \langle U \rangle_{f_1}$ , which is the most natural candidate to describe the average position of a particle on a circle:

$$\frac{\langle U \rangle_{f_\xi}}{\langle U \rangle_{f_1}} \approx e^{i\alpha}, \quad (17)$$

where the approximation is very good. We conclude that the parameter  $\alpha$  can be interpreted as a classical angle.

We now specialize to free motion in a circle. Clearly, the quantum Hamiltonian is

$$\hat{H} = \frac{1}{2} \hat{J}^2. \quad (18)$$

It turns out that as with the standard coherent states the discussed coherent states for a particle on a circle are not stable with respect to the free evolution. Namely, we have

$$Z(t)f_\xi(\varphi) = e^{-\frac{it}{2}} \xi f_\xi(\varphi + t), \quad (19)$$

where  $Z(t) = e^{it\frac{\hat{J}^2}{2}} Z e^{-it\frac{\hat{J}^2}{2}} = e^{it(\hat{J}-\frac{1}{2})} Z$ . Therefore, the problem naturally arises as to whether the wave packet  $f_\xi(\varphi)$  spreads in the circle as in the standard case of the free motion in the real line. Consider the probability distribution in the coordinate space. As time develops the wave packet for a free particle on a circle is given by

$$f_\xi(\varphi, t) = e^{-it\frac{\hat{J}^2}{2}} f_\xi(\varphi) = \theta_3\left(\frac{1}{2\pi}(\varphi - \alpha - il) \middle| \frac{1}{2\pi}(i - t)\right), \quad (\text{boson case}) \quad (20a)$$

$$f_\xi(\varphi, t) = e^{-it\frac{\hat{J}^2}{2}} f_\xi(\varphi) = \theta_2\left(\frac{1}{2\pi}(\varphi - \alpha - il) \middle| \frac{1}{2\pi}(i - t)\right). \quad (\text{fermion case}) \quad (20b)$$

It can be easily checked that both the functions (20) are  $4\pi$ -periodic. Using (20) and (15) we find that the probability density for the coordinates at time  $t$  is

$$p_{l,\alpha}(\varphi, t) = \frac{|f_\xi(\varphi, t)|^2}{\|f_\xi\|^2} = \frac{|\theta_3(\frac{1}{2\pi}(\varphi - \alpha - il) \middle| \frac{1}{2\pi}(i - t))|^2}{\theta_3(\frac{il}{\pi} \middle| \frac{i}{\pi})}, \quad (\text{boson case}) \quad (21a)$$

$$p_{l,\alpha}(\varphi, t) = \frac{|f_\xi(\varphi, t)|^2}{\|f_\xi\|^2} = \frac{|\theta_2(\frac{1}{2\pi}(\varphi - \alpha - il) \middle| \frac{1}{2\pi}(i - t))|^2}{\theta_2(\frac{il}{\pi} \middle| \frac{i}{\pi})}. \quad (\text{fermion case}) \quad (21b)$$

The above probability densities are periodic functions of time with the period  $4\pi$  (boson case) and  $2\pi$  (fermion case), respectively. It thus appears that in opposition to the standard case of a free particle on the real line the wave packets do not spread but behave rather like (linear) solitons. As a consequence of the oscillations of the probability density in the coordinate space an interesting phenomenon occurs which can be interpreted as quantum jumps on the circle. Namely, it turns out that the probability density has at  $t = t_* = (2k + 1)\pi$ , where  $k$  is integer, two identical maxima (see Fig. 1). Thus at  $t = t_*$  the particle can be detected with equal (maximal) probabilities at two different points on a circle. What about the position on a circle at  $t = t_*$ ? In order to answer this question we should first

identify a classical counterpart of the angle coordinate. As one would expect having in mind the experience with the commutator (3) the expectation value of the angle operator in the Heisenberg picture  $\hat{\varphi}(t)$  in the normalized coherent state can hardly be interpreted as such a counterpart (see Fig. 2). Instead, it appears that an appropriate candidate is the argument of the expectation value of the operator  $U(t)$  (see (7)) in the normalized coherent state  $f_\xi/\|f_\xi\|$ , so

$$\varphi_{\text{class}}(t) = \text{Arg}\langle U(t) \rangle_{f_\xi} \mod 2\pi. \quad (22)$$

Using (22) and the formula

$$\langle U(t) \rangle_{f_\xi} = e^{-\frac{1}{4}} e^{i\alpha} \frac{\theta_2(\frac{t}{2\pi} - \frac{il}{\pi} | \frac{i}{\pi})}{\theta_3(\frac{il}{\pi} | \frac{i}{\pi})}, \quad (\text{boson case}) \quad (23a)$$

$$\langle U(t) \rangle_{f_\xi} = e^{-\frac{1}{4}} e^{i\alpha} \frac{\theta_3(\frac{t}{2\pi} - \frac{il}{\pi} | \frac{i}{\pi})}{\theta_2(\frac{il}{\pi} | \frac{i}{\pi})}, \quad (\text{fermion case}) \quad (23b)$$

where  $U(t) = e^{it\frac{j^2}{2}} U e^{-it\frac{j^2}{2}}$ , we find that there is a discontinuity in the angle at  $t = t_*$  (see Fig. 3). The limit approached from the left and the limit approached from the right coincide with the abscissae of the two identical maxima of the probability density. It seems plausible to interpret such discontinuities as a manifestation of the quantum jumps on a circle. We point out that the quantum jumps take place in the boson case only for  $l$  integer and in the fermion case only for  $l$  half-integer.

Concluding, the wave packets for a particle moving freely in a circle are periodic functions of time and do not spread. As a consequence the discontinuity appears in the phase representing the position on a circle which can be regarded as quantum jumps. To our knowledge this observation provides the first example of exotic constrained dynamics of a quantum particle.

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## Figure captions

FIG. 1. The probability density  $p_{l,\alpha}$  given by (21a) (boson case), where  $\alpha = 0.75\pi$ ,  $l = 1$ , and  $t = \pi$  (solid line). The two maxima of the probability density with the abscissa  $\varphi = 2.356$  and  $\varphi = 5.498$ , respectively, are identical and equal to 1.773. For the better illustration of the evolution of the probability density the curves are placed in the figure referring to  $t = 0.9\pi$  (dotted line) and  $t = 1.1\pi$  (dash line).

FIG. 2. The evolution of the expectation value of the angle operator  $\hat{\varphi}(t)$  given by  $\langle \hat{\varphi}(t) \rangle = \frac{1}{2\pi} \int_0^{2\pi} \varphi p_{l,\alpha}(\varphi, t) d\varphi$ , where the probability density  $p_{l,\alpha}$  is given by (21a) (boson case);  $\alpha$  and  $l$  are the same as in Fig. 1. It should be noted that  $\langle \hat{\varphi}(t) \rangle$  takes the values only from the subset of the circle  $[0, 2\pi)$  given approximately by the interval (1.8, 4.8). Further, the plot is not piecewise linear as one would expect for the case of the free evolution on a circle. Such behavior of  $\langle \hat{\varphi}(t) \rangle$  indicates that it is rather poor candidate to represent the classical angle.

FIG. 3. The time dependence of the counterpart of the classical angle specified by (22) and (23a) (boson case). The values  $\text{Arg}\langle U(t) \rangle_{f_\xi} = 2.356$  and  $\text{Arg}\langle U(t) \rangle_{f_\xi} = 5.498$ , where the discontinuities appear, coincide with the abscissa  $\varphi = 2.356$  and  $\varphi = 5.498$ , respectively, of the two identical maxima of the probability density shown in Fig. 1.